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$$a^{P-1} \equiv 1 \bmod P$$

is true for every $a$ which is prime to $P$:

5.13.17
5.17.29
7.13.19
7.13.31
7.19.67
7.31.73
13.61.397
13.37.241
13.37.97
13.37.61
31.61.271
31.61.211
31.61.631
37.73.109
13.37.73.457

## ON REMARKABLE POINTS OF CURVES.

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Among the points that we define below as *remarkable*, the only ones that have been considered to any serious extent, are the well known singular points of algebraic curves. Plücker[*] in solving the famous Poncelet pardox on the class of the reciprocal of an algebraic curve, gave the formula connecting their numbers. Caily[†] subsequently gave analogous ones for twisted curves, and Veronese[‡] extended his results for $n$-space. In what follows an attempt is made to make precise the notion of remarkable point of a curve, by defining a class of points that can be reasonably so called, and two very simple propositions concerning these points are established. It is proper to remark that the discussion is not restricted to algebraic curves.

Let us consider in a plane an innumerable aggregate $A$ of points, and the innumerable aggregate $B$ of lines obtained by joining any two points of the aggregate $A$.

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[*] *Algebraische Kurven.*

[†] *Coll. Math. Papers*, Vol. 1, p. 207.

[‡] *Behandlung über die Methode des Projicirens und Schneidens, Math. Ann.* Vol. 19, p. 213.

Let $C$ be a real curve given arbitrarily in the plane, subjected to the only condition that it shall have an analytic arc. Next consider the net of curves of order $m$ passing through $\lambda$ points of $A$ and tangent to $\mu$ lines of $B$. For given values of $m$, $\lambda$, $\mu$, it is clear that we have only an innumerable aggregate of these nets, since each net is a function of a finite number of points or lines taken from a discrete aggregate of them. In fact each net can be made to correspond to a series of terms depending upon $(\lambda+\mu)$ indices, and these evidently form an innumerable aggregate. Taking a definite net there is in general a number $k$ such that at any point $M$ of $C$ there is a curve of the net having with $C$ a contact of order $k$ but none having with it a contact of order $(k+1)$. For some points of $C$, part of an innumerable aggregate, there will be at least one curve of the net having with $C$ a contact of order $(k+1)$ at these points, and also some curves of the net will have with $C$ a contact of order $k$ in more than one point. It would be conceivable that all curves of the net having a contact of order $k$ with $C$, satisfy this condition for $h$ points, in which case we would single out those having a contact of order $k$ in $(h+1)$ points. The points where there is a contact of order $(k+1)$ with a curve of the net, and the points where $C$ is touched by the curves that have $(h+1)$ times a contact of order $k$ with it will be called *remarkable points*, and we will include in this definition, the points of $C$ corresponding to the remarkable points of its reciprocal with respect to an arbitrary conic of its plane. For given values of $(m, \lambda, \mu)$ it is clear that the points in question form a discrete aggregate on $C$, and therefore the same is true when $m$, $\lambda$, $\mu$ take all integral values. Hence, *the remarkable points of a curve form a discrete aggregate on it.*

Let $O$ be an arbitrary point in the plane of $C$, and join to it all remarkable points of $C$, thus obtaining an innumerable aggregate of rays. As the lines going through $O$ form an aggregate having the power of the continuum, if we subtract from it the above discrete aggregate of rays, the remaining one has still the power of the continuum.* Hence, *the aggregate of lines cutting a curve in none of its remarkable points has the power of the continuum.* This proposition would clearly be unmodified if instead of straight lines we took any algebraic curve.

The remarkable points as above defined include evidently a great variety of points seldom associated. Thus vertices of a conic are points of hyperosculation for the circles of curvature of the curve. There the aggregate $A$ is formed of the two circular points at infinity. A multiple point is the reciprocal of points of contact of a multiple tangent of the reciprocal polar of $C$, and therefore is also included. In this case $A=B=0$; a multiple point is a remarkable point independent of the aggregate $A$.

Finally, we may say that the extension to $n$-space varieties is quite evident.

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* E. Borel, *Lecons sur la Theorie des Fonctions*, p. 13.